336 KSME Journal, Vol. 9, No. 3, pp. 336~343, 1995

On the Evaluation of Coupling Effect in Nonclassically Damped Linear Systems

Sungwon Kim*

(Received December 17, 1994)

The normal coordinates of a nonclassically damped systems are coupled by nonzero off-diagonal elements of the modal damping matrix. In analyzing a nonclassically damped system, one common approximation is to neglect those damping terms which are nonclassical, and retain the classical ones. The sufficient conditions justifying the neglect of the off-diagonal terms are that the damping matrix be diagonally dominant and the minimum frequency separation be sufficiently large. But the problem is that the amount of sufficiently small and sufficiently large is not well defined, thus decoupling approximation technique can hardly be applied to the real problems. The purpose of this paper is to provide the method for evaluating the coupling effect of nonclassically damped systems. This method does not require matrix inversion nor exact solution but simply provide a convenient way to determine the feasibility of neglecting off-diagonal elements of the modal damping matrix.

Key Words: Nonclassically Damped Systems, Diagonally Dominant, Frequency Separation, Decoupling Approximation, Coupling Effect

1. Introduction

The method of modal superposition is a very powerful technique for evaluating the response of a linear dynamic system. A linear system is said to have classical normal modes if the system possesses a complete set of real, orthonormal eigenvectors. In general, an undamped dynamic system always possesses classical normal modes. When dissipative forces are present, the system may or may not possess classical normal modes. If it does, the system is said to be classically damped. Caughey and O'Kelly(1965) established a necessary and sufficient condition for the existence of classical normal modes in a damped linear system. If classical normal modes exist, the differential equations of motion become decoupled when expressed in modal coordinates which are real. Otherwise, the system is said to be nonclassically damped. In reality, nonclassical damping comes from drastic variations of energy absorption rates of the materials in different parts of the structure. Typical examples of nonclassically damped systems are a nuclear reactor containment vessel founded on soft soil subjected to earthquake motion(Clough and Mojtaheoi, 1976), and a base-isolated structure in the same environment(Tsai and Kelly, 1988).

When dissipative forces are nonclassical, it is generally difficult to analyze the system dynamics, owing to the complex nature of the eigensolutions. Foss(1958) and Vigneron (1986) proposed a state-space approach which takes into account the orthogonality relations between the complex eigenvectors of a nonclassically damped system. The key to the utility of the eigensolution is, of course, orthogonality, which allows decoupling of the governing equations. One disadvantage of such exact methods is that they require significant numerical effort to determine the eigensolutions. The effort required is evidently intensified by the fact that the eigensolutions of a nonclassically damped system are complex. From the analysts' viewpoint, another disadvantage is the lack of physical insight afforded by the methods which

* Department of Mechanical Engineering, Korea University

are intrinsically numerical in nature. Several authors have studied nonclassically damped linear systems by approximate techniques. For instance, Cronin(1976) obtained an approximate solution for a nonclassically damped system under harmonic excitation by perturbation techniques. Using the frequency domain approach, Hasselman(1976) proposed a criterion for determining whether the equations of motion might be considered practically decoupled if nonclassical damping exists. A similar criterion was also suggested by Warburton and Soni(1977). Chung and Lee(1986) applied perturbation techniques to obtain the eigensolutions of damped systems with weakly nonclassical damping. Prater and Singh(1986) developed several indices to determine quantitatively the extent of nonclassical damping in discrete vibratory systems. Ozguven(1982, 1987) used the concept of receptance to study nonclassically damped structures in the frequency domain. Bellos and Inman(1990) examined modal coupling in linear systems by employing nonproportionality indices. A survey of the various methods that are applicable to nonclassically damped systems was recently given by Bellos and Inman(1989).

In analyzing a nonclassically damped system, one common approximation is to neglect those damping terms which are nonclassical, and retain the classical ones. This approach is termed the method of decoupling approximation. For large-scale systems, the computational effort at adopting decoupling approximation is at least an order of magnitude smaller than the method of complex modes. A whole body of literature has been devoted to this rather appealing method of approximate solution. Clearly, decoupling approximation is valid if modal coupling can somehow be ignored.

The sufficient conditions justifying the neglect of the off-diagonal terms are that modal damping matrix be diagonally dominant and minimum frequency separation be sufficiently large. If the off-diagonal terms in the modal damping matrix are sufficiently small, then they may be neglected provided the minimum frequency separation is large. But the problem is that the amount of sufficiently small and sufficiently large is not well defined, thus explaining the apparent discrepancies pointed out by I.W. Park *et al*(1992). They demonstrate the interplay between diagonal dominance and modal coupling. They have shown that neither frequency separation of the natural modes nor strongly diagonal dominance of the modal damping matrix would be sufficient to suppress the sometimes significant effect of modal coupling. However, to calculate the coupling effect, the true solutions of the response must be known, which is not practical. Also, the sufficient conditions for neglecting the off-diagonal elements can not be applied to the real problem since the term "sufficiently" is not well defined. Therefore, decoupling approximation technique requires a method to check whether the coupling effect is small or large. But an attempt to evaluate the coupling effect simply without calculating matrix inversion or true solutions has not been studied in the open literature.

The purpose of this paper is to provide the simple way of evaluating the coupling effect in nonclassically damped linear systems under harmonic excitation. The organization of the paper is as follows. In Section 2, the method of evaluating the coupling effect is explained. The method will provide a simple way of determining the feasibility of neglecting the off-diagonal elements of the modal damping matrix. Specially, interesting examples are provided in Section 3 to apply the proposed simple method for evaluating the coupling effect. In Section 4, a summary of the findings is provided.

2. Evaluation of Coupling Effect

The equation of motion of an n-degree-of-freedom system under external excitation can be written as

$$M\ddot{x}+C\dot{x}+Kx=F(t),\quad x(0)=x_0,\ \dot{x}(0)=\dot{x}_0,\ t\geq 0 \tag{1}$$

where the mass matrix M, the damping matrix C, and the stiffness matrix K are of order $n\times n$. The displacement $x(t)$ and external excitation $F(t)$ are n-dimensional vectors. For classical systems, it is assumed that M is symmetric and positive definite and C, K are symmetric and positive semi-definite. These assumptions are not arbitrary, but in fact have solid footing in the theory of Lagrangian dynamics. Symmetry of M results naturally from the transformation from Cartesian to generalized coordinates for a scleronomic system, and the positive definiteness requirement is a property of kinetic energy. Symmetry of K results from linearization of the potential energy function about equilibrium point, and the form of the Rayleigh dissipation function ensures symmetry of C. In the last several decades, passive systems have constituted a subject for intense investigation in structural mechanics.

Let U denote the $n\times n$ modal matrix associated with system (1). The modal matrix is a nonsingular matrix whose columns are eigenvectors of the symmetric eigenvalue problem

$$Ku^{(i)}=\omega_i^2 Mu^{(i)} \tag{2}$$

where $\omega_i^2>0$ and $u^{(i)}$, $i=1, \cdots, n$ are the eigenvalues and eigenvectors, respectively. The modal matrix is usually orthonormalized according to $U^TMU=I_n$, where U^T demotes the transpose of U, and I_n is the identity matrix of order n. Hence, $U^TKU=diag(\omega_1^2, \cdots, \omega_n^2)=\Omega$. . It is well-known that by the linear transformation $x(t)=Uq(t)$, Eq. (1) can be written in the normalized form

$$\begin{aligned}&\ddot{q}+D\dot{q}+\Omega q=g(t),\\ &q(0)=U^TMx_0,\ \dot{q}(0)=U^TM\dot{x}_0,\quad t\geq 0\end{aligned} \tag{3}$$

where $D=U^TCU$ and $g(t)=U^TF(t)$, and $q(t)$ is the n-dimensional vector of normal coordinates. The symmetric matrix D is called the modal damping matrix. If D is diagonal, the system is said to be classically or proportionally damped. It has shown by Caughey and O' Kelly(1965) that system (1) is classically damped if and only if $CM^{-1}K=KM^{-1}C$. In general, this condition is not satisfied, and the modal damping matrix D is not diagonal. The normalized equations of motion are then coupled by the nonzero off-diagonal elements of D, and the system is said to be nonclassically or nonproportionally damped. It is with this type of linear systems that the present paper is concerned.

Consider system (3) subjected to harmonic excitation. Let

$$g(t)=bu(t) \tag{4}$$

where $b=(b_1, b_2, \cdots, b_n)^T$, and b_i are nonzero constants for $i=1, \cdots, n$. The scalar $u(t)$ is a harmonic function of unit amplitude. Since the main focus of this study is the calculation of coupling effect in the steady-state solution, it may be assumed, for convenience, that $x_0=\dot{x}_0=0$ From the system (3), the equation of motion corresponding to the i-th mode is

$$\begin{aligned}&\ddot{q}_i+d_{ii}\dot{q}_i+\omega_i^2 q_i\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}\dot{q}_j=b_iu(t),\\ &q_i(0)=\dot{q}_i(0)=0,\quad t\geq 0\end{aligned} \tag{5}$$

where d_{ij} are the off-diagonal elements of the modal damping matrix D. Taking the Laplace transform of Eq. (5) with lower limit 0^-, we have

$$\begin{aligned}&\left[s^2+\left(d_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}\frac{\hat{q}_j(s)}{\hat{q}_i(s)}\right)s+\omega_i^2\right]\hat{q}_i(s)\\ &=b_i\hat{u}(s)\end{aligned} \tag{6}$$

where $\hat{q}_i(s)$, $\hat{q}_j(s)$ and $\hat{u}(s)$ are the Laplace transforms of $q_i(t)$, $q_j(t)$ and $u(t)$, respectively. Let the modal coupling vector $\hat{\nu}_i(s)$ for the i-th mode

$$\hat{\nu}_i(s)=\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}\frac{\hat{q}_j(s)}{\hat{q}_i(s)}. \tag{7}$$

The modal coupling vector $\hat{\nu}_i(s)$ can also be thought of as a modification made to the diagonal damping element d_{ii} to account for the coupling effect. The extent of the modification depends on the relative orientation of modes in the complex plane. Although Eq. (7) provides a qualitative indication of the modal coupling, a quantitative estimation requires further effort. If Eq. (5) were completely uncoupled, it could be written in the familiar form

$$\begin{aligned}&\ddot{q}_i^f+2\omega_i\mu_i\dot{q}_i^f+\omega_i^2 q_i^f=b_iu(t),\\ &q_i^f(0)=\dot{q}_i^f(0)=0,\quad t\geq 0\end{aligned} \tag{8}$$

where superscript f means fictitious. Here, however, the coefficients μ_i, $i=1, 2, \cdots, n$, are fictitious damping ratios calculated by using either experimentally obtained data, or convenient analytical formulas provided in the literature(Duncan and Taylor, 1979, Seireg and Haward, 1989). These ratios depend on the frequency as well as the diagonal and off-diagonal entries of the modal damping matrix. The modal damping ratios based on the approximation of neglecting the off-diagonal entries of matrix D are calculated from

$$\zeta_i=\frac{d_{ii}}{2\omega_i},\ i=1,\ 2,\ \cdots,\ n. \tag{9}$$

The fictitious damping ratios can be written(Belloi and Inman, 1980)

$$\mu_i(s)=\left|\frac{s^2+2\zeta_i\omega_i s+\omega_i^2}{2\omega_i}\sum_{j=1}^{n}\frac{d_{ij}}{s^2+2\zeta_j\omega_j s+\omega_j^2}\right| \tag{10}$$

for $i=1,\ 2,\ \cdots,\ n$, and it yields the exact modal damping ratios for proportionally damped system.

Taking the Laplace transform of Eq. (8), we have

$$\hat{q}_i^f(s)=\frac{b_i\hat{u}(s)}{s^2+2\mu_i\omega_i s+\omega_i^2} \tag{11}$$

where $\hat{q}_i^f(s)$ is the Laplace transform of $q_i^f(s)$. The transformed response of the modal coordinate i divided by the transformed response of the modal coordinate j, defines the coupling ratios in the Laplace domain. Motivated by procedures in experimental modal analysis, these ratios can be approximated by the coupling ratios of the fictitious modal responses(Bellos and Inman, 1990). Then

$$\frac{\hat{q}_j(s)}{\hat{q}_i(s)}\approx\frac{\hat{q}_j^f(s)}{\hat{q}_i^f(s)}=\frac{s^2+2\mu_i\omega_i s+\omega_i^2}{s^2+2\mu_j\omega_j s+\omega_j^2}\cdot\frac{b_j}{b_i}. \tag{12}$$

With the above expressions, the equivalent damping $\hat{e}_i(s)=d_{ii}+\hat{\nu}_i(s)$ in the Eq. (6) can be written as

$$\hat{e}_i(s)=d_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n}d_{ij}\frac{s^2+2\mu_i\omega_i s+\omega_i^2}{s^2+2\mu_j\omega_j s+\omega_j^2}\cdot\frac{b_j}{b_i}. \tag{13}$$

There have been attempts reported in the open literature to devise and utilize indices similar to the modal coupling as indicators of the extent of coupling effect. It should be addressed, however, that the concept of modal coupling does not provide a direct measure of the effect of modal coupling on the system response. For instance, a large modal coupling vector does not imply correspondingly significant coupling effect on the response of the system. A different formulation that allows a true measurement of the influence of modal coupling on the system response is needed. To this end, observe that

$$\hat{q}_i(s)=\left\{\frac{1}{s^2+\hat{e}_i(s)s+\omega_i^2}\right\}b_i\hat{u}(s). \tag{14}$$

The expression $[s^2+\hat{e}_i(s)s+\omega_i^2]^{-1}$ can be viewed as a transfer function for the modal response $\hat{q}_i(s)$. Writing j for i in the above and substituting into Eq. (6) with Eq. (14), we obtain

$$\hat{q}_i(s)=\frac{b_i\hat{u}(s)}{s^2+d_{ii}s+\omega_i^2}\left[1-\sum_{\substack{j=1\\ j\neq i}}^{n}d_{ij}\frac{s}{s^2+\hat{e}_j(s)s+\omega_j^2}\cdot\frac{b_j}{b_i}\right]. \tag{15}$$

The coupling vector $\hat{y}_i(s)$ for the i-th mode is given by

$$\hat{y}_i(s)=-\sum_{\substack{j=1\\ j\neq i}}^{n}d_{ij}\frac{b_j}{b_i}\frac{s}{s^2+\hat{e}_j(s)s+\omega_j^2}. \tag{16}$$

As in the case of the modal coupling vector, $\hat{y}_i(s)$ is the vector sum of its elements in the complex plane. From Eq. (15), it is easy to see that $\hat{y}_i(s)$ can be viewed as a correction factor to the modal transfer function. In contrast to modal coupling, a large coupling vector always indicates correspondingly significant modal coupling effect on the response of the system. Therefore, the simple procedure for evaluating the coupling effect is to calculate the Eqs. (9), (10), (13) and (16).

In a system with a large number of degrees of freedom, there are many ways that modal coupling effect can become appreciable. The overall effect of coupling, however, basically depends on the coupling vector $\hat{y}_i$ defined in Eq. (16). The coupling vector $\hat{y}_i$ directly measures the influence of modal coupling on the response of the system. In the complex plane, the magnitude of $\hat{y}_i$ depends not only on the magnitudes of its elements, but also on how their directions are aligned in the complex plane. Even when the individual elements are small, their sum can be large if they possess similar phase angles. On the other hand, summing relatively large elements may result in a coupling vector with negligible magnitude if there are cancellations among a significant number of them. Therefore, predicting the extent of modal coupling without an examination of the coupling vectors may lead to erroneous conclusions.

3. Applications to Multi-Degree-of-Freedom Systems

The normalized coordinates of a nonclassically damped system are coupled by the non-zero off-diagonal elements of the damping matrix. One common procedure in the solution of a nonclassically damped system is to neglect the off-diagonal elements of the normalized damping matrix. For a multi-degree-of-freedom system, substantial reduction in computational effort is achieved by this method of decoupling the system. But, whenever the approximation introduced by disregarding the off-diagonal elements is applied, the extent of approximation should be checked easily in advance. In this paper, the simple method for evaluating the overall coupling effect has been proposed in the previous section, and it is applied to the following example. Although a limited set of cases is presented, extensive calculations have been performed, and all the calculations have yielded identical qualitative results on the coupling effect.

Example. Consider a six-degree-of-freedom system whose normalized equation of motion is

$$\ddot{q} + D\dot{q} + \Omega q = b\sin(\omega_f t),$$
$$q(0) = \dot{q}(0) = 0, \ t \geq 0. \tag{17}$$

In order to apply the proposed method for evaluating the coupling effect, the following four interesting cases are provided.

Case (1) :

$$D = \begin{bmatrix} 1.61 & 0.5 & 0.3 & -0.5 & 0.0 & -0.3 \\ 0.5 & 1.7 & 0.0 & -0.3 & -0.5 & 0.3 \\ 0.3 & 0.0 & 1.8 & 0.5 & -0.3 & -0.5 \\ -0.5 & -0.3 & 0.5 & 1.75 & 0.3 & 0.0 \\ 0.0 & -0.5 & -0.3 & 0.3 & 1.65 & 0.5 \\ -0.3 & 0.3 & -0.5 & 0.0 & 0.5 & 1.8 \end{bmatrix} \tag{18}$$

$$\Omega = diag(3.95^2\ 3.98^2\ 4.0^2\ 4.1^2\ 4.11^2\ 4.25^2) \tag{19}$$

$$\omega_f = 1.0,\ 3.8,\ 3.95,\ 4.0,\ 4.11,\ 4.25,\ 4.3,\ 7.0,\ 8.0 \tag{20}$$

$$b = (1,\ 1,\ 1,\ 1,\ 1,\ 1)^T \tag{21}$$

The normalized damping matrix is weakly diagonally-dominant and frequency separation is small. It looks like the coupling effect is large. But when applying the proposed method to this case, it is easy to see that the coupling effect is very small even at the frequency where the excitation is the same with the natural modes, as shown in Fig. 1. Therefore, decoupling approximation can be used safely in this case.

Figures 2 and 3 represent the effect of input vectors. Cases 1~6 are classified by increasing the magnitude from 1 to 6th element in the input vector twice. If the excitation is far from the narrowly separated natural modes($\omega_f = 8.0$), the effect of input vector is appreciable as shown in Fig. 2. But as can be seen in Fig. 3, the effect of input vector is very large on the effect of modal coupling when the excitation is near the narrowly separated natural modes($\omega_f = 4.0$).

Case (2) :

$$D = D \ in \ equation \ (18)$$

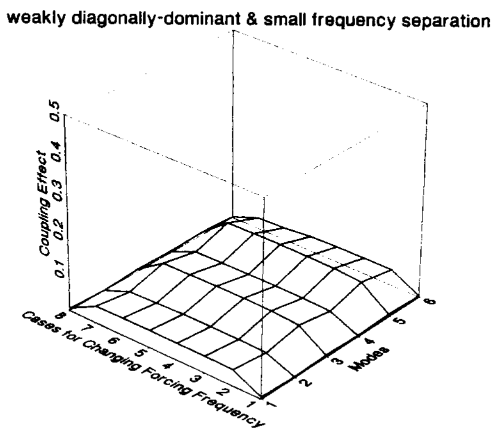

Fig. 1 Coupling effect for Case (1)

$$\Omega = diag(1.9^2\ 2.8^2\ 3.5^2\ 4.0^2\ 4.8^2\ 5.3^2) \quad (22)$$

$$\omega_f = 0.2,\ 1.0,\ 1.9,\ 2.8,\ 3.5,\ 4.0,\ 4.8,\ 5.3,\ 5.5,\ 8.0 \quad (23)$$

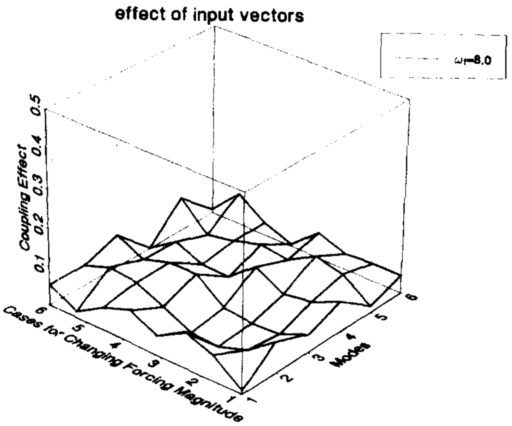

Fig. 2 Coupling effect for Case (1)

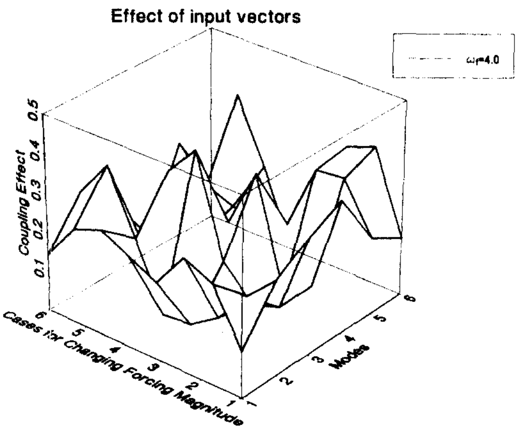

Fig. 3 Coupling effect for Case (1)

Here, the frequency separation between the natural modes is large. This may lead to erroneous prediction that the coupling effect is small. But, on the contrary, the coupling effect is very large at the natural modes in Fig. 4. Nevertheless, as can be seen in Figs. 1 and 4, the coupling effect is suppressed significantly if the frequency separation between the excitation and the natural modes is large.

Case (3) :

$$D = \begin{bmatrix} 1.61 & 0.06 & 0.08 & 0.07 & 0.05 & 0.04 \\ 0.06 & 1.70 & 0.02 & 0.05 & 0.06 & 0.03 \\ 0.08 & 0.02 & 1.80 & 0.03 & 0.07 & 0.05 \\ 0.07 & 0.05 & 0.03 & 1.75 & 0.04 & 0.02 \\ 0.05 & 0.06 & 0.07 & 0.04 & 1.65 & 0.05 \\ 0.04 & 0.03 & 0.05 & 0.02 & 0.05 & 1.80 \end{bmatrix} \quad (24)$$

$\Omega = \Omega$ *in equation* (19)

$\omega_f = \omega_f$ *in equation* (20)

In this case, the modal damping matrix is strongly diagonally-dominant and the frequency separation between the natural modes is small. Since the modal damping matrix is strongly diagonally-dominant, it seems that the coupling effect is smaller than the Case (1). However, the coupling effect is larger than the Case (1) as shown in Fig. 5.

Case (4) :

$D = D$ *in equation* (24)

$\Omega = \Omega$ *in equation* (22)

$\omega_f = \omega_f$ *in equation* (23)

Here, the modal damping matrix is strongly

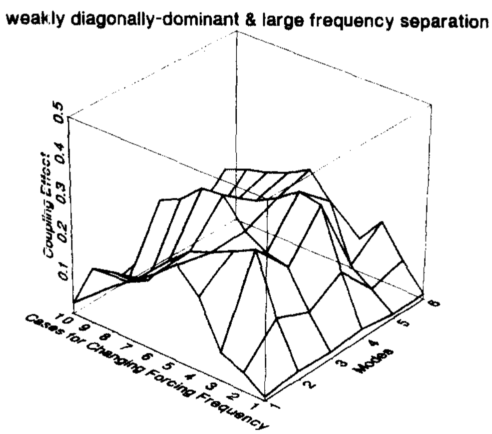

Fig. 4 Coupling effect for Case (2)

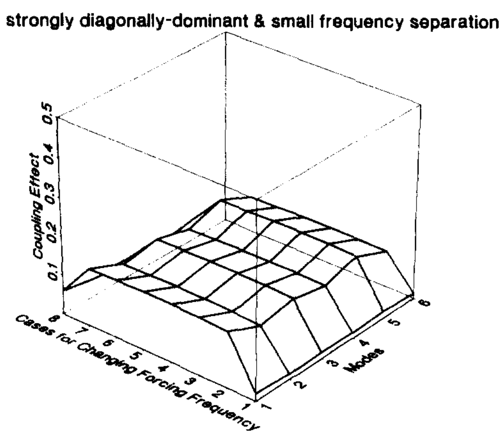

Fig. 5 Coupling effect for Case (3)

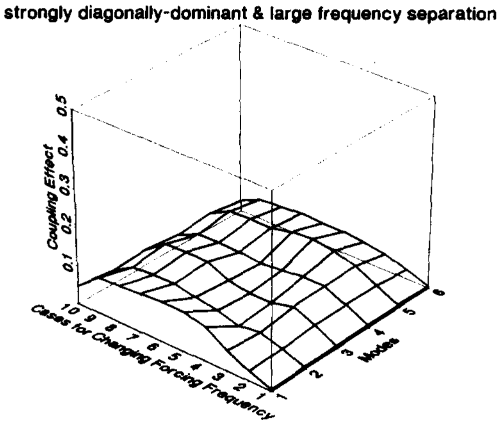

Fig. 6 Coupling effect for Case (4)

diagonally-dominant and the frequency separation between the natural modes is large. As expected, the coupling effect is decreased significantly compared to the Case (3) as in Fig. 5.

From the above four cases, it can be concluded that neither the diagonal dominance nor the frequency separation between the natural modes would suppress the coupling effect linearly. And the extent of modal coupling is primarily determined by the frequency separation between the excitation and the natural modes. Therefore, predicting the feasibility of decoupling approximation by the properties of the diagonal dominance and frequency separation between the natural modes will lead to erroneous result. The extent of modal coupling can be determined from the proposed method of evaluating the coupling vectors given in Eq. (16).

6. Conclusions

The normal coordinates of a nonclassically damped system are coupled by nonzero off-diagonal elements of the modal damping matrix. In this paper, a simple method of evaluating the modal coupling effect arising from these off-diagonal elements has been investigated. In the process, the proposed method for evaluating the coupling vector has also been applied to an example of 6 degree-of-freedom system. The major results are summarized in the following.

(1) The sufficient conditions justifying the neglect of off-diagonal terms in the modal damping matrix are that the matrix be diagonally dominant and the minimum frequency separation be sufficiently large. However, the term "sufficiently" is not well defined, thus decoupling approximation technique should be applied to the real-world problems with an extreme caution.

(2) Neither the diagonal dominance nor the frequency separation between the natural modes would suppress the coupling effect linearly.

(3) The extent of modal coupling is primarily determined by the frequency separation between the excitation and the natural modes.

(4) The simple method for evaluating the coupling effect of a nonclassically damped system is proposed in this paper.

(5) The extent of modal coupling can be determined from the suggested simple way of evaluating coupling vectors given by Eq. (16).

It is believed that the above results are very useful in the field of nonclassical systems, and from the proposed method the feasibility of decoupling approximation can be checked easily.

References

Bellos, J. and Inman, D. J., 1989, "A Survey of Nonproportional Damping," *Shock and Vibration Digest*, Vol. 21, pp. 7~12.

Bellos, J. and Inman, D. J., 1990, "Frequency Response of Nonproportionally Damped, Lumped Parameter, Linear Dynamic Systems," *ASME J. Vibration and Acoustics*, Vol. 112, pp. 194~201.

Caughey, T. K. and O'Kelly, M. E. J., 1965, "Classical Normal Modes in Damped Linear Dynamic Systems," *ASME J. Appl. Mech.*, Vol. 32, pp. 583~588.

Chung, K. R. and Lee, C. W., 1986, "Dynamic Reanalysis of Weakly Nonproportionally Damped Systems," *J. Sound and Vibration*, Vol. 111, pp. 37~50.

Clough, R. W. and Mojtahedi, S., 1976, "Earthquake Response Analysis Considering Non-Proportional Damping," *Earthq. Engng. Dyn.*, Vol. 4, pp. 489~496.

Cronin, D. L., 1976, "Approximation for Determining Harmonically Excited Response of Nonclassically Damped Systems," *ASME J. Engineering for Industry*, Vol. 98, pp. 43～47.

Duncan, P. E. and Taylor, R. E., 1979, "A Note on the Dynamic Analysis of Nonproportionally Damped Systems," *Earthq. Engng. Struct. Dyn.*, Vol. 7, pp. 99～105.

Foss, K. A., 1958, "Co-Ordinates which Uncouple the Equations of Motion of Damped Linear Dynamic System," *ASME J. Appl. Mech.*, Vol. 25, pp. 361～364.

Hasselman, T. K., 1976, "Modal Coupling in Lightly Damped Structures," *AIAA J.*, Vol. 14, pp. 1627～1628.

Nair, S. S. and Singh, R., 1986, "Examination of the Validity of Proportional Damping Approximations with Two Further Numerical Indices," *J. Sound and Vibration*, Vol. 104, pp. 348～350.

Ozguven, H. N., 1982, "Receptances of Non-Proportionally and Continuously Damped Plates-Reduced Dampers Method," *J. Sound and Vibration*, Vol. 85, pp. 383～395.

Ozguven, H. N., 1987, "On the Harmonic Response of Non-Proportionally Damped Structures," *Proc. 11th ASME Biennial Conference on Mechanical Vibration and Noise*, Vol. 5, pp. 117～124.

Park, I. W., Kim, J. S. and Ma, F., 1992, "On Modal Coupling in Non-Classically Damped Linear Systems," *Mechanics Research Communications*, Vol. 19(5), pp. 407～413.

Prater, Jr. G. and Singh, R., 1986, "Quantification of the Extent of Non-Proportional Viscous Damping in Discrete Vibratory Systems," *J. Sound and Vibration*, Vol. 104, pp. 109～125.

Seireg, A. and Haward, L., 1967, "An Approximate Normal Mode Method for Damped Lumped Parameter Systems," *ASME J. Eng. Industry*, pp. 597～604.

Tsai, H.-C. and Kelly, J.M., 1988, "Non-Classical Damping in Dynamic Analysis of Base-Isolated Structures with Internal Equipment," *Earthq. Engng. Struct. Dyn.*, Vol. 16, pp. 29～43.

Vigneron, F. R., 1986, "A Natural Modes Model and Modal Identities for Damped Linear Structures," *ASME J. Appl. Mech.*, Vol. 53, pp. 33～38.

Warburton, G. B. and Soni, S. R., 1977, "Errors in Response Calculations for Non-Classically Damped Structures," *Earthq. Engng. Struct. Dyn.*, Vol. 5, pp. 365～376.